

# Transient analysis of a $M/M/\infty$ queue with discouragement and for the related embedded chain

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## Abstract

Consider the following birth and death process with the following infinitesimal transition probabilities  $\lambda_k = \frac{\lambda}{1+k}$  and  $\mu_k = \mu k$  with  $\lambda, \mu > 0$ . This process has known as a *discouragement queue* [5].

Although from the theoretical point of view the problem of the determination of the transition functions has been solved [4], the explicit form of them is not present in literature.

We have solved this problem assuming that the solution is representable by a Taylor series, under the initial condition that the process starts to state zero.

We discuss also the same problem for the embedded chain and using direct computation we obtain a recursive formula for the transient distribution.

## 1 Introduction

The problem to find an explicit formula of the transition probabilities for this particular queue is proposed by Natvig [5].

Parthasarathy et al. [6] give an explicit solution, based on continued fractions approach, for the cases  $\lambda_k = \frac{\lambda}{1+k}$  and  $\mu_k = k$  and  $\lambda_k = \lambda$  and  $\mu_k = \mu k$ , with  $\lambda, \mu > 0$ .

Despite the powerful of continued fractions approach, for our case this method seems to be fail.

We propose a more simple approach based on the Taylor expansion, and recognize a simple iterative formula for the coefficients of the series. Our approach permits a fast numerically calculation of the coefficients.

Unexpectedly the bound we used for the coefficients, to verify the total convergence of the Taylor series, is related with the Bessel Numbers [2].

## 2 Transient solution in continuous time

We remember that that the infinitesimal transition probabilities are given by

$$P(X(t+h) = n+m | X(t) = n) = \begin{cases} \frac{\lambda}{1+k}h + o(h), & \text{if } m = 1 \\ o(h), & \text{if } |m| > 1 \\ \mu kh + o(h), & \text{if } m = -1 \end{cases}$$

Let  $q(k, t) = P(X(t) = k)$  then

$$\begin{cases} q'(0, t) = -\lambda q(0, t) + \mu q(1, t) \\ q'(k, t) = \frac{\lambda}{k} q(k-1, t) - \left( \frac{\lambda}{1+k} + k\mu \right) q(k, t) + (k+1)\mu q(k+1, t), \quad \forall k \geq 1 \end{cases} \quad (1)$$

Let  $\tau = \lambda t$ ,  $p(k, \tau) = q(k, t(\tau))$ ,  $\alpha = \sqrt{\frac{\mu}{\lambda}}$  and  $\dot{p}(k, \tau) = \frac{1}{\lambda} \dot{q}(k, t)$ , then (15) is equivalent to

$$\begin{cases} \dot{p}(0, \tau) = -p(0, \tau) + \alpha^2 p(1, \tau) \\ \dot{p}(k, \tau) = \frac{1}{k} p(k-1, \tau) - \left( \frac{1}{1+k} + k\alpha^2 \right) p(k, \tau) + (k+1)\alpha^2 p(k+1, \tau), \quad \forall k \geq 1 \end{cases} \quad (2)$$

We consider (2) under the initial conditions

$$p(0, 0) = 1, \quad p(k, 0) = 0 \quad \forall k > 0$$

Let us observe that the functions  $p(k, \cdot)$  have derivatives of any orders. This observations suggest a Taylor series expansion for  $p(k, \tau)$ ,  $k \geq 0$ , and in particular for  $p(0, \tau)$ .

For sake of simplicity we put  $p(k, \tau) = \frac{w(k, \tau)}{\alpha^k k!}$ , then the  $p(k, \tau)$ ,  $\tau \geq 0$ , satisfy (2) if and only if the  $w(k, \tau)$ ,  $k \geq 0$ , satisfy

$$\begin{cases} \dot{w}(0, \tau) = -w(0, \tau) + \alpha w(1, \tau) \\ \dot{w}(k, \tau) = \alpha w(k-1, \tau) - \left( \frac{1}{1+k} + k\alpha^2 \right) w(k, \tau) + \alpha w(k+1, \tau), \quad \forall k \geq 1 \end{cases} \quad (3)$$

with the initial conditions  $w(0, 0) = 1$ ,  $w(k, 0) = 0$ .

Let  $b_k = \frac{1}{k+1} + k\alpha^2$ ,  $k \geq 0$ , we observe that if

$$w(0, \tau) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} r_i, \quad r_0 = 1 \quad (4)$$

then

$$\begin{aligned} w(1, \tau) &= \frac{1}{\alpha} \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} [r_{i+1} + b_0 r_i]; \\ w(2, \tau) &= \frac{1}{\alpha^2} \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} [r_{i+2} + b_0 r_{i+1} + b_1 (r_{i+1} + b_0 r_i) - \alpha^2 r_i]; \\ w(3, \tau) &= \frac{1}{\alpha^3} \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} \{ (r_{i+3} + b_0 r_{i+2}) + b_1 (r_{i+2} + b_0 r_{i+1}) - \alpha^2 r_{i+1} + \\ &\quad + b_2 [r_{i+2} + b_0 r_{i+1} + b_1 (r_{i+1} + b_0 r_i) - \alpha^2 r_i] - \alpha^2 (r_{i+1} + b_0 r_i) \}; \\ &\dots \end{aligned}$$

So that if we put  $S_i^{(0)} = r_i$ ,  $S_i^{(-1)} = 0$  for all  $i \geq 0$  and  $S_i^{(n)} = 0$  for all  $n > i$ , by recursion we have

$$\begin{cases} S_i^{(1)} = S_{i+1}^{(0)} + b_0 S_i^{(0)} \\ S_i^{(k+1)} = S_{i+1}^{(k)} + b_k S_i^{(k)} - \alpha^2 S_i^{(k-1)}, \quad \forall i \geq 0, \quad 0 \leq k \leq i+1 \end{cases} \quad (5)$$

so that

$$w(k, \tau) = \frac{1}{\alpha^k} \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} S_i^{(k)}, \quad \forall k \geq 0 \quad (6)$$

and obviously

$$p(k, \tau) = \frac{1}{\alpha^{2k} k!} \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} S_i^{(k)}, \quad \forall k \geq 0 \quad (7)$$

**Proposition 2.1.** *Let  $S_0^{(0)} = 1$  and  $S_i^{(-1)} = 1$ , for all  $i \geq 0$ , then (5) is equivalent to*

$$\begin{cases} S_k^{(k)} = 1, \quad \forall k \geq 0 \\ S_{h+k+1}^{(k)} = \alpha^2 S_{k+h}^{(k-1)} - b_k S_{k+h}^{(k)} + S_{k+h}^{(k+1)}, \quad \forall h, k \geq 0 \end{cases} \quad (8)$$

furthermore

$$S_{k+h+1}^{(k)} = \sum_{i=0}^k \alpha^{2(k-i)} \left[ -b_i S_{i+h}^{(i)} + S_{i+h}^{(i+1)} \right], \quad h, k \geq 0 \quad (9)$$

**Proof**

(8) is obvious. To prove (9) we proceed by induction.

Thesis is true for  $n = 0$  because

$$S_{h+1}^{(0)} = \alpha^2 S_h^{(-1)} - b_0 S_h^{(0)} + S_h^{(1)} = \sum_{i=0}^0 \alpha^{-2i} \left[ -b_i S_{i+h}^{(i)} + S_{i+h}^{(i+1)} \right]$$

then supposing it is true for  $k > 0$  we have

$$\begin{aligned}
S_{(k+1)+h+1}^{(k+1)} &= \alpha^2 S_{k+1+h}^{(k+1-1)} - b_{k+1} S_{k+1+h}^{(k+1)} + S_{k+1+h}^{(k+1+1)} = \\
&= \alpha^2 S_{k+1+h}^{(k)} - b_{k+1} S_{k+1+h}^{(k+1)} + S_{k+1+h}^{(k+2)} = \\
&= \alpha^2 \sum_{i=0}^k \alpha^{2(k-i)} \left[ -b_i S_{i+h}^{(i)} + S_{i+h}^{(i+1)} \right] + \left[ -b_{k+1} S_{k+1+h}^{(k+1)} + S_{k+1+h}^{(k+2)} \right] = \\
&= \sum_{i=0}^k \alpha^{2(k+1-i)} \left[ -b_i S_{i+h}^{(i)} + S_{i+h}^{(i+1)} \right] + \alpha^{2(k+1-(k+1))} \left[ -b_{k+1} S_{k+1+h}^{(k+1)} + S_{k+1+h}^{((k+1)+1)} \right] = \\
&= \sum_{i=0}^{k+1} \alpha^{2(k+1-i)} \left[ -b_i S_{i+h}^{(i)} + S_{i+h}^{(i+1)} \right]
\end{aligned}$$

□

To establish the convergence of the Taylor series (7) we observe that if we put

$$L_i^{(k)} \doteq \frac{(-1)^{i-k}}{\alpha^{2k}} S_i^{(k)} \quad (10)$$

from (5) we have

$$L_{i+1}^{(k)} = L_i^{(k-1)} + b_k L_i^{(k)} + \alpha^2 L_i^{(k+1)}, \quad \forall i \geq 0, \quad 0 \leq k \leq i+1$$

and

$$L_0^{(-1)} = 0, \quad \forall i \geq 0; \quad L_i^{(k)} = 0, \quad \forall k > i; \quad L_i^{(i)} = 1, \quad \forall i \geq 0$$

It is obvious that such  $L_i^{(k)}$  are non negative for all  $i, k = 0, 1, 2, \dots$  and moreover if we put  $\gamma = \max(1, \alpha^2)$  we have

$$L_i^{(k)} \leq M_i^{(k)} \gamma^{2(i-k)}, \quad \forall k, i = 0, 1, 2, \dots \quad (11)$$

where the  $M_i^{(k)}$  are such that

$$\begin{cases} M_{i+1}^{(k)} = M_i^{(k-1)} + (1+k)M_i^{(k)} + M_i^{(k+1)}, & \forall i \geq 0, \quad 0 \leq k \leq i+1 \\ M_0^{(-1)} = 0, & \forall i \geq 0; \quad M_i^{(k)} = 0, \quad \forall k > i; \quad M_i^{(i)} = 1, \quad \forall i \geq 0 \end{cases} \quad (12)$$

Computing the  $M_i^{(k)}$ , we obtain that

$$M_i^{(0)} = B_i^*, \quad \forall i \geq 0$$

where the  $B_i^*$  are the so called Bessel numbers. This last assertion derives from the comparison of the generating function related to the solution of the equations (12), see [1] for more details (paragraph 7.3), and the generating function of the Bessel Number proposed in [2].

In [2] it is proved that Bessel numbers have the asymptotic form

$$B_i^* \sim \frac{1}{\sqrt{2\pi i}} \frac{w^{i+3}}{(w!)^2}, \quad \forall i \geq 0$$

where  $w \sim \frac{i}{2 \ln(i)}$  is the positive root of following equation:

$$i + 2 = 2w \ln(w)$$

Then, apart from sub-exponential factors, Bessel numbers grown like

$$B_i^* \approx \left( \frac{i}{2e \ln(i)} \right)^i \quad (13)$$

From (13) it follows that the power series

$$\sum_{i=0}^{+\infty} (-1)^i \frac{\tau^i}{i!} M_i^{(0)} \quad (14)$$

has an infinite convergence radius.

Therefore taking into account (11) by the convergence of (14), we conclude that the series at the second member of (6) is a power series with infinite convergence radius.

In conclusion we have established the following theorem.

**Theorem 2.2.** *The Cauchy problem (4) has a unique solution  $p(k, \tau)$ ,  $k = 0, 1, 2, \dots$ , where  $p(k, \tau)$ ,  $\tau \geq 0$ , are defined in (7).*

### 3 Embedded chain

We start with the general state dependent case and we recover the solution of our problem as corollary of the general case.

We consider now the following birth and death process with infinitesimal transition probabilities

$$P(X(t+h) = k+m | X(t) = k) = \begin{cases} \lambda_k h + o(h), & \text{if } m = 1 \\ o(h), & \text{if } |m| > 1 \\ \mu_k h + o(h), & \text{if } m = -1 \end{cases}$$

Let  $p(k, t) = P(X(t) = k)$  then

$$\begin{cases} p'(0, t) = -\lambda p(0, t) + \mu p(1, t) \\ p'(k, t) = \lambda_{k-1} p(k-1, t) - (\lambda_k + \mu_k) p(k, t) + \mu_{k+1} p(k+1, t), \quad \forall k \geq 1 \end{cases} \quad (15)$$

**Definition 3.1.** A stochastic process  $\{X(t)\}_{t \geq 0}$  taking its values in the countable state space  $E$  is called a jump process if for the almost  $\omega \in \Omega$  and all  $t \geq 0$ , there exists  $\epsilon(t, \omega) > 0$  such that

$$X(t+s, \omega) = X(t, \omega), \quad \forall s \in [t, t + \epsilon(t, \omega))$$

It is called regular jump process if in addition, for almost all  $\omega \in \Omega$ , the set  $A(\omega)$  of discontinuities of the function  $t \rightarrow X(t, \omega)$  is  $\sigma$ -discrete, that is, for all  $c \geq 0$

$$|A(\omega) \cap [0, c]| < +\infty$$

where the notation  $|B|$  is the cardinality of set  $B$ . A regular jump homogeneous Markov chain is by definition a continuous time **HMC** that is also regular jump process.

Let  $\{\tau_n\}$  be a non decreasing sequence of transition times of the regular jump process  $\{X(t)\}_{t \geq 0}$  where  $\tau_0 = 0$  and  $\tau_n = \infty$  if there are strictly fewer than  $n$  transitions in  $(0, \infty)$ .

The process  $\{X_n\}_{n \geq 0}$  with value in  $E_\Delta = E \cup \Delta$ , where  $\Delta$  is an arbitrary element not in  $E$ , is defined by  $X_n = X(\tau_n)$  with the convention  $X(\infty) = \Delta$ , and it is called *embedded process* of the jump process.

The associated embedded process, see [3], has transition probabilities of the form

$$\begin{cases} p_{i,j} = \frac{\lambda_i}{\lambda_i + \mu_i}, & j = i+1, \quad i \geq 1 \\ p_{i,j} = \frac{\mu_i}{\lambda_i + \mu_i}, & j = i-1, \quad i \geq 1 \\ p_{i,j} = 1, & j = 1, \quad i = 0 \\ p_{i,j} = 0 & \text{otherwise} \end{cases}$$

Now consider the associated embedded processes  $\{X_n\}_{n \geq 0}$  of (15).

Let  $P(X_n = k) = p_{n,k}$  then

$$\begin{cases} p_{n+1,k} = \frac{\lambda_{k-1}}{\lambda_{k-1} + \mu_{k-1}} p_{n,k-1} + \frac{\mu_{k+1}}{\lambda_{k+1} + \mu_{k+1}} p_{n,k+1}, & \forall n \geq 0, \quad 1 \leq k \leq n+1 \\ p_{n,k} = 0, & \forall k > n \\ p_{n,-1} = 0, & \forall n \geq 0 \\ p_{0,0} = 1 \end{cases} \quad (16)$$

The next two following lemma are elementary and they will be explained without proofs.

**Lemma 3.2.** Let  $(p_{n,k})_{n,k \geq 0}$  be the matrix defined in (16) then

$$(i) \quad p_{n,n} = \prod_{i=0}^n \left( \frac{\lambda_i}{\lambda_i + \mu_i} \right), \quad \forall n > 0$$

$$(ii) \quad \sum_{k=0}^{+\infty} p_{n,k} = 1, \quad \forall n \geq 0$$

From (16) it follows obviously the following lemma.

**Lemma 3.3.** In the hypothesis of lemma (3.2), if  $n + k$  is odd then  $p_{n,k} = 0$ .

By virtue of Lemma (3.2) and Lemma (3.3) we can find the transient distribution.

**Proposition 3.4.** Let  $(p_{n,k})_{n,k \geq 0}$  as in (16) then

$$\begin{cases} p_{n,k} = \prod_{i=0}^k \left( \frac{\lambda_i}{\lambda_i + \mu_i} \right) T_{\frac{n-k}{2},k}, & n \geq 0, \quad 0 \leq k \leq n, \quad n+k \text{ even} \\ p_{n,k} = 0, & \text{otherwise} \end{cases} \quad (17)$$

where

$$T_{h,k} = \sum_{l=0}^k \frac{\lambda_l}{\lambda_l + \mu_l} \left( 1 - \frac{\lambda_{l+1}}{\lambda_{l+1} + \mu_{l+1}} \right) T_{h-1,l+1}, \quad \forall k \geq 0, h \geq 1; \quad T_{0,k} = 1, \quad \forall k \geq 0$$

**Proof**

We know from the lemma (3.3) that if  $n + k$  is odd then  $p_{n,k} = 0$ , furthermore it holds that

$$d_k := p_{k,k} = \prod_{i=0}^k \left( \frac{\lambda_i}{\lambda_i + \mu_i} \right) \quad (18)$$

Let  $n = k + 2h$ , we want prove that

$$p_{k+2h,k} = d_k T_{h,k}, \quad \forall k \geq 0, h \geq 0 \quad (19)$$

Fixed  $k$ , we proceed by induction on  $h$ . The (19) is true for  $h = 0$  being

$$p_{k,k} = d_k = d_k T_{0,k}$$

supposing it is true for  $h$  we prove it for  $h + 1$ . Let  $\alpha_k := \frac{\lambda_k}{\lambda_k + \mu_k}$  from (16) we have

$$\begin{aligned} p_{k+2(h+1),k} &= p_{k+2h+1,k-1} \alpha_{k-1} + p_{k+2h+1,k+1} \alpha_{k+1} = \\ &= p_{k+2h+1,k-1} \alpha_{k-1} + [1 - \alpha_{k+1}] p_{k+2h+1,k+1} \end{aligned}$$

and by the inductive hypothesis (19), it follows that

$$p_{k+2(h+1),k} = p_{k+2h+1,k-1}\alpha_{k-1} + [1 - \alpha_{k+1}]d_{k+1}T_{h,k+1}$$

Taking into account (18) we have

$$p_{k+2(h+1),k} = p_{k+2h+1,k-1}\alpha_{k-1} + (d_{k+1} - d_{k+2})T_{h,k+1} \quad (20)$$

On other hand for every fixed  $h$ , setting

$$b_k^h = p_{k+2h+1,k-1}\alpha_{k-1} \quad (21)$$

we have (to show)

$$\begin{cases} b_0^h = 0 \\ b_k^h = d_k \sum_{i=0}^{k-1} \frac{d_{i+1} - d_{i+2}}{d_i} T_{h,i+1}, \quad \forall k \geq 1 \end{cases} \quad (22)$$

In fact proceeding by induction also in this case, we have that (22) is true for  $k = 0$ , being  $p_{2h+1,-1} = 0$ . Furthermore it is true for  $k = 1$  because for (21) it holds that

$$b_1^h = p_{2+2h,0}$$

and from (16) we have

$$b_1^h = p_{1+2h,-1} + (1 - \alpha_1)p_{1+2h,1} = (1 - \alpha_1)p_{1+2h,1}$$

thus by inductive hypothesis on  $h$  it follows that

$$b_1^h = (1 - \alpha_1)d_1T_{h,1} = (d_1 - d_2)T_{h,1} = d_1 \frac{d_1 - d_2}{d_0} T_{h,1}$$

being  $d_0 = d_1 = 1$ .

Now supposing that (22) is true for  $k \geq 1$ , we prove it for  $k + 1$ :

$$\begin{aligned} b_{k+1}^h &= p_{k+1+2h+1,k}\alpha_k = \\ &= \alpha_k [p_{k+1+2h,k}\alpha_k + (1 - \alpha_{k+1})p_{k+1+2h,k+1}] \end{aligned}$$

for the (21) and by inductive hypothesis on  $h$  (19) we obtain

$$b_{k+1}^h = \alpha_k [b_k^h + (1 - \alpha_{k+1})d_{k+1}T_{h,k+1}]$$

furthermore by inductive hypothesis on  $k$  (22) we have



$$\begin{aligned}
b_{k+1}^h &= \alpha_k \left[ d_k \sum_{i=0}^{k-1} \frac{d_{i+1} - d_{i+2}}{d_i} T_{h,i+1} + (1 - \alpha_{k+1}) d_{k+1} T_{h,k+1} \right] = \\
&= \alpha_k \left[ d_k \sum_{i=0}^{k-1} \frac{d_{i+1} - d_{i+2}}{d_i} T_{h,i+1} + (d_{k+1} - d_{k+2}) T_{h,k+1} \right] = \\
&= \alpha_k d_k \sum_{i=0}^k \frac{d_{i+1} - d_{i+2}}{d_i} T_{h,i+1} = \\
&= d_{k+1} \sum_{i=0}^k \frac{d_{i+1} - d_{i+2}}{d_i} T_{h,i+1}
\end{aligned}$$

Now for  $h$  fixed the (21) is true, substituting (21) in (20) finally we obtain

$$\begin{aligned}
p_{k+2(h+1),k} &= d_{k+1} \sum_{i=0}^k \frac{d_{i+1} - d_{i+2}}{d_i} T_{h,i+1} + (d_{k+1} - d_{k+2}) T_{h,k+1} = \\
&= d_{k+1} \left[ \sum_{i=0}^k \frac{d_{i+1} - d_{i+2}}{d_i} T_{h,i+1} + \frac{d_{k+1} - d_{k+2}}{d_{k+1}} T_{h,k+1} \right] = \\
&= d_{k+1} \sum_{i=0}^{k+1} \frac{d_{i+1} - d_{i+2}}{d_i} T_{i+1}^{(h)} = d_{k+1} T_{h+1,k}
\end{aligned}$$

then the thesis follows. □

As an immediate corollary of the proposition 3.4 we have the following:

**Corollary 3.5.** *Let  $(p_{n,k})_{n,k \geq 0}$  as in (16) with  $\lambda_k = \frac{\lambda}{1+k}$  and  $\mu_k = \mu k 1_{\{k \geq 1\}}$  with  $\lambda, \mu > 0$ , then*

$$\begin{cases} p_{n,k} = d_k T_k^{(\frac{n-k}{2})}, & n \geq 0, \quad 0 \leq k \leq n, \quad n+k \text{ even} \\ p_{n,k} = 0, & \text{otherwise} \end{cases} \quad (23)$$

where

$$\begin{aligned}
d_k &= \prod_{i=0}^k \frac{1}{1 + i(i-1)\alpha^2} \\
T_k^{(h)} &= \sum_{i=0}^k \frac{d_{i+1} - d_{i+2}}{d_i} T_{i+1}^{(h-1)}, \quad \forall k \geq 0, h \geq 1; \quad T_k^{(0)} = 1, \quad \forall k \geq 0
\end{aligned}$$

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